ARTÍCULO ORIGINAL

THERE IS NO GROUP OF REAL GENUS 72

No hay grupos de género real 72

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RESUMEN

Todo grupo finito G actúa como grupo de automorfismos de diversas superficies de Klein con borde. Al menor de los géneros algebraicos de estas superficies se le llama género real $\rho(G)$ del grupo G. Se sabe que todos los enteros positivos impares son género real de algún grupo. En cambio, no todos los pares lo son. C. L. May recopiló una serie de familias de grupos, a partir de las cuales obtiene sucesiones aritméticas de géneros reales pares con las que se cubre gran parte del conjunto de los números pares. En particular, resultó que los números 2, 12 y 24 no son género real de ningún grupo. May se planteó si este era un fenómeno propio de números pequeños, o bien existen otros huecos en el espectro del género real, esto es, números N que no son el género real de ningún grupo. El primer número para el que esta cuestión permanecía abierta es 72. En el presente trabajo probamos que no existe ningún grupo de género real 72.

PALABRAS CLAVE: Género real, superficies de Klein, grupos de automorfismos, grupos NEC.

ABSTRACT

Every finite group G acts as an automorphism group of several bordered Klein surfaces. The minimal genus of these surfaces is called the real genus $\rho(G)$ of the group G. It is known that all odd positive integers are the real genus of some group. On the contrary, not all even integers are. C. L. May compiled a series of families of groups, from which he obtained arithmetic sequences of even numbers which are real genus of some group, covering a large part of the even numbers. In particular, it results that 2, 12 and 24 are not the real genus of any group. May asked on whether this is as a question of small numbers, or else there are other gaps in the spectrum of the real genus, that is to say, numbers N such that there are no groups of real genus N. The first number on which the question remained unsolved is 72. In the present paper we prove that there is no group of real genus 72.

KEYWORDS: Real genus, Klein surfaces, automorphism groups, NEC groups.

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1. INTRODUCTION

A Klein surface is a compact surface S endowed with a dianalytic structure. A particular case consists of orientable unbordered Klein surfaces, which are the classical Riemann surfaces. Let S be a Klein surface of topological genus g with k boundary components. The algebraic genus of S is $p = \eta g + k - 1$, where $\eta = 2$ if the surface is orientable, and $\eta = 1$ otherwise.

A non-Euclidean crystallographic (NEC) group is a discrete subgroup of the group of isometries of the hyperbolic plane $\mathcal{D}$, with compact quotient space $\mathcal{D}/\Gamma$. NEC groups play an important role since both Klein surfaces and their automorphism groups are characterized by means of them.

Let S be a Klein surface of algebraic genus $p \geq 2$, topological genus g, and k boundary components. Then, there exists an NEC group Γ, called surface group, such that S and $\mathcal{D}/\Gamma$ are isomorphic.

A finite group G is an automorphism group of the surface S if and only if there exists an NEC group Λ such that Γ is a normal subgroup of Λ, and there exists an epimorphism $\theta\colon \Lambda \to G$ with $\ker(\theta) = \Gamma$, such that $G = \frac{\Lambda}{\Gamma}$.

Every finite group G acts as automorphism group of different Riemann and Klein surfaces. This arises the problem of determining the minimal genius of the surfaces on which the group G acts. Taking into account non-orientable unbordered Klein, all finite group G is an automorphism group of some of them, and the minimal among their topological genus is called the symmetric crosscap number of G, denoted by $\tilde{\sigma}(G)$. This parameter will be useful in our work, which concerns bordered Klein surfaces, either orientable or not. Since also every finite group G acts on some bordered surfaces, their minimal algebraic genus is called the real genus of G, denoted by $\rho(G)$. The present work deals with this parameter.

When studying the real genus, a number of problems arise. The most important are the following:

1. Take a family of finite groups, and obtain their real genus.
2. To obtain the real genus of groups up to a given order.
3. Given a certain ρ, find the groups of real genus ρ.
4. Determine which integers are the real genus of some group.

Here we are concerned with the last problem, and we will use the results known on the others.

The real genus was defined by Coy L. May, who started its systematic study. The real genus of all groups of order lesser than 64 is currently known. These results were obtained as follows: for $o(G) < 16$ by C.L. May [15]; $16 \leq o(G) < 24$ by C.L. May [17]; $24 \leq o(G) < 32$ by C.L. May [16]; $o(G) = 32$ by C.L. May [20]; $32 < o(G) < 48$ by C.L. May [21]; $o(G) = 48$ by J.J. Etayo and E. Martínez [9]; and for $48 < o(G) < 64$, by C. Cano [5].

Also, all groups satisfying $\rho(G) \leq 16$ have been determined, as well as the groups with $\rho(G) = 18, 20, 22, 24$. This was obtained for $\rho(G) \leq 3$ by C.L. May [15]; $\rho(G) = 4$ by C.L. May [14]; $\rho(G) = 5$ by C.L. May [16] and [19]; $\rho(G) = 6, 7, 8$ by G. Gromadzki and B. Mockiewicz [11], and J.J. Etayo and E. Martínez [8]; $\rho(G) = 12$ by B. Mockiewicz [23]; and $9 \leq \rho(G) \leq 16$, as well as $\rho(G) = 18, 20, 22, 24$, by C.L. May [21].

In order to determine which numbers are the real genus of some group, C.L. May proved in [18] that the dicyclic group DC_n of order $4n$ has real genus $2n + 1$. Hence all odd numbers are the real genus of some group, and the problem reduces to even numbers.

It is known that there are not groups of real genus 2, 12 or 24. For genus 2, this was obtained by E. Bujalance and J.M. Gamboa [3] earlier than the definition of real genus. The real genus 12 was studied by B. Mockiewicz [23], and the genus 24 by C.L. May [21]. C.L. May himself obtained in [20] a number of arithmetic sequences of even numbers which are the real genus of some group. The first numbers not included in them are 72, 84, 108, 168 and 192, see [24] for all the list up to 10.000. So, these are the first numbers on which the question on whether they are the real genus of some group is not solved. The goal of the present work is to prove that in fact 72 is not the real genus of any group.

2. PRELIMINARIES

The main tool in order to study bordered Klein surfaces and their automorphism groups are the non-Euclidean crystallographic (in short, NEC) groups.

Definition. A non-Euclidean crystallographic (NEC) group Γ is a discrete subgroup of the group of isometries of the hyperbolic plane $\mathcal{D}$, with compact quotient space $\mathcal{D}/\Gamma$.

H.C. Wilkie obtained in [27] a presentation of the NEC group Γ by generators and relations as follows:

Generators:

$x_i,\ i = 1, \ldots, r$	(elliptic elements)
$e_i,\ i = 1, \ldots, k$	(orientable elements, usually hyperbolic)
$c_{i,j},\ i = 1, \ldots, k,\ j = 0, \ldots, s_i$	(reflections)
$a_i,\ b_i,\ \mathrm{i} = 1, \ldots, g$ (*)	(hyperbolic) or
$d_i,\ \mathrm{i} = 1, \ldots, g$ (**)	(glide-reflections)

These generators satisfy the following relations:

$x_i^{m_i} = 1,\ i = 1, \dots, r;$

${c_{i,j-1}}^2 = {c_{i,j}}^2 = \left(c_{i,j-1}c_{i,j}\right)^{n_{i,j}} = 1,\ i = 1, \dots, k,\ j = 1, \dots, s_i;$

${e_i}^{-1}c_{i,0}e_i c_{i,s_i} = 1,\ i = 1, \dots, k;$

$x_1 \dots x_r e_1 \dots e_k a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = 1$, in case (*), or

$x_1 \dots x_r e_1 \dots e_k d_1^2 \dots d_g^2 = 1$, in case (**).

This presentation was summarized by A.M. Macbeath in [12] in a symbol called signature.

Definition. An NEC signature is a set constituted by

(i) an integer $g \geq 0$,

(ii) a sign $+$ or -,

(iii) a set of integers $m_1, \dots, m_r,\ \ m_i \geq 2, r \geq\ 0$, called proper periods, and

(iv) a set of ordered sets of integers

$$\{C_1 = \left(n_{1,1}, \dots, n_{1,s_1}\right), \dots, C_k = \left(n_{k,1}, \dots, n_{k,s_k}\right)\}, k \geq 0, n_{i,j} \geq 2, s_i \geq 0,$$

called period-cycles. The numbers $n_{i,j}$ are called link-periods.

The presentation indicated with (*) corresponds to the sign $+$ in the signature, and case (**) with the sign -.

This signature is denoted as $(g; \pm; [m_1, \dots, m_r]; \{C_1, \dots, C_k\})$ or more explicitely

$$\left(g; \pm; [m_1, \dots, m_r]; \{\left(n_{1,1}, \dots, n_{1,s_1}\right), \dots, \left(n_{k,1}, \dots, n_{k,s_k}\right)\}\right).$$

If it has not proper periods we write $[-]$, if there are not period-cycles we denote by $\{-\}$, and the empty period-cycles, with $s_i = 0$, by $(-)$.

Definition. We call surface NEC groups those having signature $\left(g; \pm; [-]; \{(-)^k\}\right)$. If $k > 0$, we call it a bordered surface group.

The signature of the NEC group Γ determines also the area of its fundamental regions. We call it the area of Γ, $|\Gamma|$, and using Gauss-Bonnet theorem, D. Singerman obtained in [26]

Theorem. Let Γ be an NEC group with signature

$$\left(g; \pm; [m_1, \dots, m_r]; \{\left(n_{1,1}, \dots, n_{1,s_1}\right), \dots, \left(n_{k,1}, \dots, n_{k,s_k}\right)\}\right).$$

Then

$$|\Gamma| = 2\pi\left\{\eta g + k - 2 + \sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\left(1-\frac{1}{n_{i,j}}\right)\right\}$$

where $\eta =$ 2 or 1 if the sign of the signature is + or - respectively.

A necessary and sufficient condition for the existence of an NEC group Γ is that the area $|\Gamma|$ be positive.

The subgroups of finite index of an NEC group are also NEC groups, and they satisfy the Riemann-Hurwitz formula. Therefore, if Γ is a subgroup of an NEC group Γ' with index N, then the relation between their areas is

$$[\Gamma' : \Gamma] = \frac{|\Gamma|}{|\Gamma'|},$$

or in other words,

$$|\Gamma| = N \cdot |\Gamma'|$$

Klein surfaces and their automorphism groups are characterized by NEC groups by means of the following results. First, R. Preston in [25] and D. Singerman [26] proved:

Theorem. Let S be a Klein surface of algebraic genus $p \geq 2$, topological genus g, and k boundary components. Then there exists a surface NEC group Γ, with signature

$$(g; \pm; [-]; \{(-), \overset{k}{\ldots}, (-)\}),$$

and sign "+" or "-" according to whether the surface is orientable or not, such that S and $\mathcal{D}/\Gamma$ are isomorphic as Klein surfaces.

And second, C.L. May proved in [13]:

Theorem. A finite group G is an automorphism group of a Klein surface $S = \mathcal{D}/\Gamma$ if and only if there exists an NEC group Λ and an epimorphism $\theta : \Lambda \to G$ such that $\ker(\theta) = \Gamma$, and $G = \Lambda/\Gamma$.

Then, let a finite group G be an automorphism group of the surface $S = \mathcal{D}/\Gamma$. Using Riemann-Hurwitz formula, we have that

$$\mathrm{o}(G) \cdot |\Lambda| = |\Gamma| = 2\pi(\eta g + k - 2).$$

Dividing by 2π and calling the expression $|\Lambda|^* = \frac{|\Lambda|}{2\pi}$ the reduced area of Λ, we obtain

$$\mathrm{o}(G) \cdot |\Lambda|^* = \eta g + k - 2 = p - 1,$$

and so

$$p = 1 + \mathrm{o}(G) \cdot |\Lambda|^*,$$

where p is the algebraic genus of the Klein surface S.

In order to obtain the real genus of a finite group G we must find the minimal algebraic genus of the bordered Klein surfaces on which G acts as an automorphism group. This is equivalent to find the minimal reduced area of an NEC group Λ admitting bordered surface groups as normal subgroups, and such that one can define the epimorphism $\theta : \Lambda \to G$.

When looking for the NEC group Λ having a bordered surface group as subgroup, we will need the following result obtained by E. Bujalance and E. Martínez in [4]:

Lemma. An NEC group Λ has a bordered surface subgroup if and only if the signature of Λ has an empty period-cycle, or a period-cycle with two consecutive link-periods equal to 2.

In order to obtain the real genus of a finite group G, there are two steps. The first one is to find an NEC group Λ_0 admitting a surface group as a normal subgroup, and to construct an epimorphism $\theta : \Lambda_0 \to G$ such that the generators of G are obtained by means of the images of the generators of Λ_0, in such a way that $\ker(\theta) = \Gamma$. We obtain so an upper bound for the real genus of G, because $\rho(G) \leq p = 1 + \mathrm{o}(G) \cdot |\Lambda_0|^*$. The second step is to prove that there exists no NEC group Λ with reduced area lesser than $|\Lambda_0|^*$, such that there exists an epimorphism $\theta : \Lambda \to G$.

3. LOOKING FOR GROUPS OF REAL GENUS 72

We start here the search for groups of real genus 72. Firstly, we are going to see a number of restrictions on the possible orders of G, and on the signatures of Λ. We are looking for groups of order N, having real genus 72. First of all, the real genus of all groups of order up to 63 is known, and none of them is 72. Hence we restrict ourselves to $N \geq 64$. On the other hand, C.L. May proved in [22] that if $p \equiv 11 \pmod{12}$ is prime, then the groups of real genus $p + 1$ have order at most $3p$; and hence, in our case $N \leq 213$.

Now, we recall that the real genus of some families of groups is already known. The cyclic and dihedral groups, C_n and D_n, have real genus 0. The groups $C_2 \times C_n$ with $n \geq 4$ even, and $C_2 \times D_n$ with n even, have real genus 1. Finally, the dicyclic groups DC_n have real genus $2n + 1$. Hence none of those groups has real genus 72. We use these results to eliminate possible orders N, as follows. Let $p < q$ be odd primes.

- If $N = p$, the unique group of order N is the cyclic group C_p.
- If $N = 2p$, the groups of order N are the cyclic group C_{2p} and the dihedral group D_p.
- If $N = 4p$ with $p \equiv 3 \pmod 4$, the groups of order N are the cyclic group C_{4p}, the dihedral group D_{2p}, the dicyclic group DC_p and the group $C_2 \times C_{2p}$.
- If $N = pq$, with $q \not\equiv 1 \pmod p$, the unique group of order N is the cyclic group C_{pq}.
- If $N = pq$, with $q \equiv 1 \pmod p$, the groups of order N are the cyclic group C_{pq} and a non-Abelian group G_{pq} which has real genus $q(p-2)+1 \neq 72$.

The symmetric crosscap number, $\tilde{\sigma}(G)$, allows to eliminate other values of N. It was proved in [1] that if the order of G is odd, then $\tilde{\sigma}(G) = \rho(G)+1$. Since M.D.E. Conder obtained in [6] the symmetric crosscap number of all groups of order up to 127, and no such group of odd order has $\tilde{\sigma}(G) = 73$, we can also eliminate the odd values $N \leq 127$.

We consider now the signatures for Λ. Since there must exist an epimorphism $\theta : \Lambda \to G$ whose kernel is a surface group without elements of finite order but reflections, the proper periods m_i and the link-periods $n_{i,j}$ in the signature of Λ must divide N. But we have more restrictions, as follows:

- If N is a proper period, then Λ has an element of order N, and so has G. Then G would be cyclic.
- If N or $N/2$ is a link-period, then Λ has two elements of order 2, with product of order N or $N/2$. The same happens in G, and so G is cyclic or dihedral.
- If N/t, for t odd, is a link-period, then the group G of order N contains the dihedral group $D_{N/t}$, but the order of this one, $2N/t$, does not divide N.

We are ready to look for all possible signatures, corresponding to groups of order N, with $64 \leq N \leq 213$, with the restrictions indicated above both for N and for the signature. The result is the following table including 33 signatures, for 14 different orders. For each N, the NEC group Λ must have reduced area $|\Lambda|^*$.

$o(G)$	$\|\Lambda\|^*$		Signature
66	71/66		$(0,+,[2,3,11],\{(-)\})$
72	71/72	i)	$(0,+,[-],\{(2,2,2,3,9,12)\})$
72	71/72	ii)	$(0,+,[-],\{(2,2,2,4,6,9)\})$
72	71/72	iii)	$(0,+,[-],\{(2,2,3,3,4,9)\})$
72	71/72	iv)	$(0,+,[3],\{(2,2,4,9)\})$
80	71/80	i)	$(0,+,[-],\{(2,2,8,20,20)\})$
80	71/80	ii)	$(0,+,[-],\{(2,2,2,2,8,10)\})$
80	71/80	iii)	$(0,+,[-],\{(-),(8,10)\})$
80	71/80	iv)	$(0,+,[2],\{(2,2,8,10)\})$
80	71/80	v)	$(0,+,[16],\{(2,2,10)\})$
80	71/80	vi)	$(0,+,[20],\{(2,2,8)\})$
80	71/80	vii)	$(0,+,[16,20],\{(-)\})$
84	71/84	i)	$(0,+,[-],\{(2,2,6,14,14)\})$
84	71/84	ii)	$(0,+,[-],\{(2,2,2,2,6,7)\})$
84	71/84	iii)	$(0,+,[-],\{(2,2,2,3,3,7)\})$
84	71/84	iv)	$(0,+,[-],\{(-),(6,7)\})$
84	71/84	v)	$(0,+,[2],\{(2,2,6,7)\})$
84	71/84	vi)	$(0,+,[3],\{(2,2,2,7)\})$
84	71/84	vii)	$(0,+,[12],\{(2,2,7)\})$
84	71/84	viii)	$(0,+,[14],\{(2,2,6)\})$
84	71/84	ix)	$(0,+,[12,14],\{(-)\})$
88	71/88		$(0,+,[-],\{(2,2,4,11,22)\})$
90	71/90		$(0,+,[9,10],\{(-)\})$
96	71/96		$(0,+,[-],\{(2,2,3,8,16)\})$
100	71/100	i)	$(0,+,[-],\{(2,2,2,25,25)\})$
100	71/100	ii)	$(0,+,[25],\{(2,2,2)\})$
100	71/100	iii)	$(0,+,[4,25],\{(-)\})$
120	71/120		$(0,+,[-],\{(2,2,2,4,15)\})$
132	71/132		$(0,+,[-],\{(2,2,2,3,11)\})$
160	71/160		$(0,+,[-],\{(2,2,16,20)\})$
168	71/168		$(0,+,[-],\{(2,2,12,14)\})$
180	71/180		$(0,+,[-],\{(2,2,9,10)\})$
200	71/200		$(0,+,[-],\{(2,2,4,25)\})$

Observe that reordering the link-periods in a signature gives rise to non-isomorphic NEC groups. Nevertheless, since we will argue just on which numbers are proper periods and link-periods, we have included in the table just one signature for each set of numbers.

Now, we analyze all 33 cases arising from the table. For each order, and each signature, we take into account that if a proper period m appears in the signature, the group G must contain a subgroup isomorphic to C_m; and if the signature has a link-period n, then the group G contains a subgroup isomorphic to D_n. So, with the help of the SmallGroupsLibrary (see [10]), we look for groups of order N containing the adequate subgroups. In the notation of that library the i-th group of order N is denoted as $[N, i]$.

In what follows we do that search for each of the fourteen values of N. In many cases we find groups of the form $C_m \times D_n$, or $D_m \times D_n$. Their real genus has been determined by A. Bacelo et al. in [2], and we will use those results. Also, we use the symmetric crosscap number of some groups, which was obtained by M.D.E. Conder in [6].

$$N = 66$$

There are just four groups of order 66. They are C_{66} and D_{33}, which have real genus 0, and $C_{11} \times D_3$ and $C_3 \times D_{11}$, which have real genus 23.

$$N = 72$$

Signature i). Then, G must contain D_9 and D_{12}. The only group is D_{36}, of real genus 0.

Signatures ii), iii) and iv). The group G must contain both D_4 and D_9. There are three groups of order 72 in those conditions. The first one is D_{36} which has real genus 0. The two others are [72, 8] and [72, 15].

We consider now $G = [72, 8]$, which has structure $(C_{18} \times C_2) \rtimes C_2$. This group has symmetric crosscap number 20, as an image of an NEC group with signature $(0, +, [2,4], \{(-)\})$. This implies that G is generated by three elements of orders 2, 2 and 4, say A, B, C, and there exists an epimorphism

$$(0, +, [2,4], \{(-)\}) \rightarrow G$$

defined by

$$\varphi(x_1) = A,\ \varphi(x_2) = C,\ \varphi(e_1) = (AC)^{-1},\ \varphi\left(c_{1,0}\right) = B.$$

Then, we take an NEC group Λ with signature $(0, +, [2,2,4], \{(-)\})$ and define the epimorphism

$$\theta :\ \Lambda = (0, +, [2,2,4], \{(-)\}) \rightarrow G$$

by

$$\theta(x_1) = A,\ \theta(x_2) = B,\ \theta(x_3) = C,\ \theta(e_1) = (ABC)^{-1},\ \theta\left(c_{1,0}\right) = 1.$$

Since $|\Lambda|^* = \frac{3}{4}$, we have that $\rho(G) \leq 1 + \frac{3}{4} \cdot 72 = 55$.

And now we deal with $G = [72,15]$. This group was studied by M.D.E. Conder and P. Dobcsanyi in [7], where it is proved that G is generated by three elements of order 2. Hence it can be obtained as an image of an NEC group Λ with signature $(0, +, [2,2,2], \{(-)\})$ and reduced area $\frac{1}{2}$. Hence, $\rho(G) \leq 1 + \frac{1}{2} \cdot 72 = 37$.

$$N = 80$$

Signature i). The group G must contain D_8 and D_{20}. The only groups are D_{40}, of real genus 0, and the group [80,15] which will be studied afterwards.

Signatures ii), iii) and iv). The group G must contain D_8 and D_{10}. The only groups are the same, D_{40}, of real genus 0, and the group [80,15].

Signature v). Then, the group must contain C_{16} and D_{10}. There is no such group of order 80.

Signature vi). The group G must contain C_{20} and D_8. The only groups are D_{40}, of real genus 0, the group [80,15], and $C_5 \times D_8$, which has real genus 31.

Signature vii). The group G must contain C_{16} and C_{20}. The only groups are C_{80}, of real genus 0, and the groups [80,1], and [80,3], which are both semidirect products $C_5 \rtimes C_{16}$. Since these two groups are generated by an element of order 5 and another one of order 16, they are images of an NEC group Λ of signature $(0, +, [5,16], \{(-)\})$, of reduced area $|\Lambda|^* = \frac{59}{80}$. Therefore, $\rho(G) \leq 1 + \frac{59}{80} \cdot 80 = 60$. In fact, [80,1] has real genus 60, see May [20].

We are only left with the group [80,15], which has structure $(C_5 \times D_4) \rtimes C_2$. Its symmetric crosscap number is 29, as an image of an NEC group with signature $(0, +, [-], \{(2,2,5,8)\})$. So, G is generated by four elements of order 2, say A, B, C, D, satisfying $(AB)^2 = (BC)^2 = (CD)^5 = (DA)^8 = 1$.

Then, we take an NEC group Λ with signature $(0, +, [-], \{(2,2,2,5,8)\})$ and define the epimorphism

$$\theta : \Lambda = (0, +, [-], \{(2,2,2,5,8)\}) \rightarrow G$$

by

$$\theta(c_{1,0}) = A, \theta(c_{1,1}) = 1, \theta(c_{1,2}) = B, \theta(c_{1,3}) = C, \theta(c_{1,4}) = D, \theta(c_{1,5}) = A.$$

Since $|\Lambda|^* = \frac{47}{80}$, we have that $\rho(G) \leq 1 + \frac{47}{80} \cdot 80 = 48$.

$$N = 84$$

Signature i). Then, G must contain D_6 and D_{14}. The only groups are D_{42}, of real genus 0, and $D_3 \times D_7$, of real genus 15.

Signatures ii), iv) and v). The group G must contain D_6 and D_7. The same groups as above appear.

Signature iii). Then, G must contain D_3 and D_7. Again the same groups appear.

Signature vi). Then, G must contain C_3 and D_7. The possible groups are D_{42}, and $D_3 \times D_7$, again, and also $C_6 \times D_7$, of real genus 29, and the group [84,7] which we study afterwards.

Signature vii). In this case, G must contain D_7 and C_{12}. There is no such group of order 84.

Signature viii). In this case, G must contain C_{14} and D_6. There are three groups, namely $D_3 \times D_7$, $C_{14} \times D_3$, which has real genus 29, and D_{42}.

Signature ix). The group G must contain C_{12} and C_{14}. Four groups are possible, which are C_{84}, of real genus 0, $C_4 \times G_{21}$, where G_{21} is the non-Abelian group of order 21, $C_3 \times DC_7$, and the group [84,1].

So, we need to study four groups, namely $C_4 \times G_{21}$, $C_3 \times DC_7$, the group [84,1], and the group [84,7].

The group $C_4 \times G_{21}$. Since G_{21} is generated by two elements of order 3, say A, B, and C_4 by an element of order 4, C, we take an NEC group NEC group Λ with signature $(0, +, [3,12], \{(-)\})$ and define the epimorphism

$$\theta : \Lambda = (0, +, [3,12], \{(-)\}) \rightarrow G$$

by

$$\theta(x_1) = A,\ \theta(x_2) = BC,\ \theta(e_1) = (ABC)^{-1},\ \theta(c_{1,0}) = 1.$$

Since $|\Lambda|^* = \frac{7}{12}$, we have that $\rho(G) \leq 1 + \frac{7}{12} \cdot 84 = 50$. In fact, $\rho(G) = 50$, see May [20].

The group $C_3 \times DC_7$. In the same way, DC_7 is generated by two elements of order 4, and C_3 by an element of order 3. So we can take an NEC group Λ with signature $(0, +, [4,12], \{(-)\})$ and reduced area $\frac{2}{3}$, and so $\rho(G) \leq 1 + \frac{2}{3} \cdot 84 = 57$. Really, $\rho(G) = 57$.

The group [84,7]. It has structure $C_2 \times (G_{21} \rtimes C_2)$, and its symmetric crosscap number is 30, as image of an NEC group with signature $(0, +, [2,6], \{(-)\})$. So, G is generated by two elements of order 2, A, B, and one of order 6, C.

Then, we take an NEC group Λ with signature $(0, +, [2,2,6], \{(-)\})$ and define the epimorphism

$$\theta : \Lambda = (0, +, [2,2,6], \{(-)\}) \rightarrow G$$

by

$$\theta(x_1) = A,\ \theta(x_2) = B,\ \theta(x_3) = C,\ \theta(e_1) = (ABC)^{-1},\ \theta(c_{1,0}) = 1.$$

Since $|\Lambda|^* = \frac{5}{6}$, we have that $\rho(G) \leq 1 + \frac{5}{6} \cdot 84 = 71$.

Finally, we consider the group [84,1]. Its structure is $DC_7 \rtimes C_3$, and it has a presentation given by generators A, B, C, of orders, respectively 4, 7, and 3, satisfying the relations

$$A^4 = B^7 = C^3 = 1, \quad ABA^{-1} = B^{-1}, \ CBC^{-1} = B^4, \ ACA^{-1} = C.$$

Since $AC = CA$, the group can be generated by the element B of order 7, and the element AC of order 12, because $(AC)^4 = C, (AC)^9 = A$. Hence we take an NEC group Λ with signature $(0, +, [7,12], \{(-)\})$ and define the epimorphism

$$\theta : \Lambda = (0, +, [7,12], \{(-)\}) \to G$$

by

$$\theta(x_1) = B, \ \theta(x_2) = AC, \ \theta(e_1) = (BAC)^{-1}, \ \theta(c_{1,0}) = 1.$$

Since $|\Lambda|^* = \frac{65}{84}$, we have that $\rho(G) \leq 1 + \frac{65}{84} \cdot 84 = 66$.

We have finished with the groups of order 84, and none of them can have real genus 72.

$$N = 88$$

The unique signature requires that the group G contains both D_4 and D_{22}. Only two groups satisfy this condition, D_{44}, of real genus 0, and the group [88,7], which has structure $(C_{22} \times C_2) \rtimes C_2$. This group has symmetric crosscap number 24, as an image, among others, of an NEC group with signature $(0, +, [-], \{(2,2,4,4)\})$. So, G is generated by four elements of order 2, say A, B, C, D, satisfying $(AB)^2 = (BC)^2 = (CD)^4 = (DA)^4 = 1$.

Then, we take an NEC group Λ with signature $(0, +, [-], \{(2,2,2,4,4)\})$ and define the epimorphism

$$\theta : \Lambda = (0, +, [-], \{(2,2,2,4,4)\}) \to G$$

by

$$\theta(c_{1,0}) = A, \theta(c_{1,1}) = 1, \theta(c_{1,2}) = B, \theta(c_{1,3}) = C, \theta(c_{1,4}) = D, \theta(c_{1,5}) = A.$$

Since $|\Lambda|^* = \frac{1}{2}$, we have that $\rho(G) \leq 1 + \frac{1}{2} \cdot 88 = 45$.

$$N = 90$$

The group must contain C_9 and C_{10}. The only groups are C_{90}, of real genus 0, and $C_5 \times D_9$, which has real genus 37.

$$N = 96$$

The unique signature requires that the group contains D_3 and D_{16}. Only two groups satisfy these conditions, D_{48}, of real genus 0, and the group [96,33]. This last group has structure $C_3 \rtimes D_{16}$. Thus, it is generated by an element of order 3 and two elements of order 2, and so it is an image of an NEC group Λ with signature $(0, +, [2,2,3], \{(-)\})$. Since $|\Lambda|^* = \frac{2}{3}$, we have that $\rho(G) \leq 1 + \frac{2}{3} \cdot 96 = 65$.

$$N = 100$$

Signature i). The group must contain $C_2 \times C_2$ and D_{25}. The unique such group is D_{50}, of real genus 0.

Signature ii). Then, G must contain C_{25} and $C_2 \times C_2$. We have only D_{50}, and $C_2 \times C_{50}$, of real genus 1.

Signature iii). The group must contain C_4 and C_{25}. There are three groups, C_{100}, of real genus 0, DC_{25}, of real genus 51, and the group [100,3].

So, we need only to consider the group [100,3], which has structure $C_{25} \rtimes C_4$. This group has symmetric crosscap number 25, as an image of an NEC group with signature $(0, +, [4], \{(25)\})$. This implies that there exists an epimorphism

$$(0, +, [4], \{(25)\}) \to G$$

defined by

$$\varphi(x_1) = A,\ \varphi(e_1) = A^{-1},\ \varphi(c_{1,0}) = B,\ \varphi(c_{1,1}) = ABA^{-1}.$$

So, G is generated by an element A of order 4, and two elements, A and ABA^{-1}, of order 2, such that $BABA^{-1}$ has order 25. Then, we take an NEC group Λ with signature $(0, +, [-], \{(-), (25)\})$ and define the epimorphism

$$\theta :\ \Lambda = (0, +, [-], \{(-), (25)\}) \to G$$

by

$$\theta(e_1) = A,\ \theta(e_2) = A^{-1},\ \theta(c_{1,0}) = 1,\ \theta(c_{2,0}) = B,\ \theta(c_{2,1}) = ABA^{-1}.$$

Since $|\Lambda|^* = \frac{12}{25}$, we have that $\rho(G) \leq 1 + \frac{12}{25} \cdot 100 = 49$.

$$N = 120$$

The unique signature requires that G contains D_4 and D_{15}. There exist five groups in these conditions. One is D_{60}, of real genus 0. We are going to study the four other groups, which are the following.

The group [120,12]. It has structure $C_3 \rtimes D_{20}$. This group has symmetric crosscap number 39, as an image of an NEC group with signature $(0, +, [-], \{(2,2,3,20)\})$. So, G is generated by four elements of order 2, say A, B, C, D, satisfying $(AB)^2 = (BC)^2 = (CD)^3 = (DA)^{20} = 1$.

Then, we take an NEC group Λ with signature $(0, +, [-], \{(2,2,2,3,20)\})$ and define the epimorphism

$$\theta :\ \Lambda = (0, +, [-], \{(2,2,2,3,20)\}) \to G$$

by

$$\theta(c_{1,0}) = A, \theta(c_{1,1}) = 1, \theta(c_{1,2}) = B, \theta(c_{1,3}) = C, \theta(c_{1,4}) = D, \theta(c_{1,5}) = A.$$

Since $|\Lambda|^* = \frac{67}{120}$, we have that $\rho(G) \leq 1 + \frac{67}{120} \cdot 120 = 68$.

The group [120,13]. This group has structure $C_3 \rtimes ((C_{10} \times C_2) \rtimes C_2)$, with symmetric crosscap number 42, as an image of an NEC group with signature $(0,+,[-],\{(2,2,4,12)\})$. As in the previous case, we take an NEC group Λ with signature $(0,+,[-],\{(2,2,2,4,12)\})$ and define the epimorphism

$$\theta : \Lambda = (0,+,[-],\{(2,2,2,4,12)\}) \to G$$

by

$$\theta(c_{1,0}) = A, \theta(c_{1,1}) = 1, \theta(c_{1,2}) = B, \theta(c_{1,3}) = C, \theta(c_{1,4}) = D, \theta(c_{1,5}) = A.$$

Since $|\Lambda|^* = \frac{7}{12}$, we have that $\rho(G) \leq 1 + \frac{7}{12} \cdot 120 = 71$.

The group [120,30]. This group has also structure $C_3 \rtimes ((C_{10} \times C_2) \rtimes C_2)$, and it has symmetric crosscap number 32, as an image of an NEC group with signature $(0,+,[-],\{(2,2,4,4)\})$. We act as before, take an NEC group Λ with signature $(0,+,[-],\{(2,2,2,4,4)\})$, and define the epimorphism

$$\theta : \Lambda = (0,+,[-],\{(2,2,2,4,4)\}) \to G$$

by

$$\theta(c_{1,0}) = A, \theta(c_{1,1}) = 1, \theta(c_{1,2}) = B, \theta(c_{1,3}) = C, \theta(c_{1,4}) = D, \theta(c_{1,5}) = A.$$

Since $|\Lambda|^* = \frac{1}{2}$, we have that $\rho(G) \leq 1 + \frac{1}{2} \cdot 120 = 61$.

And the group [120,38]. It has structure $C_5 \rtimes S_4$, and its symmetric crosscap number is 13, as an image of an NEC group with signature $(0,+,[-],\{(2,4,15)\})$. Hence, it is generated by three elements of order 2, say A, B, C, satisfying $(AB)^2 = (BC)^4 = (CA)^{15} = 1$. We now take an NEC group Λ with signature $(0,+,[-],\{(2,2,4,15)\})$, and define the epimorphism

$$\theta : \Lambda = (0,+,[-],\{(2,2,4,15)\}) \to G$$

by

$$\theta(c_{1,0}) = A,\ \theta(c_{1,1}) = 1,\ \theta(c_{1,2}) = B,\ \theta(c_{1,3}) = C,\ \theta(c_{1,4}) = A.$$

Since $|\Lambda|^* = \frac{41}{120}$, we have that $\rho(G) \leq 1 + \frac{41}{120} \cdot 120 = 42$.

$$N = 132$$

The group G must contain D_3 and D_{11}. The unique such groups are D_{66}, of real genus 0, and $D_3 \times D_{11}$, of real genus 23.

$$N = 160$$

The group must contain D_{16} and D_{20}. There are just two groups, D_{80}, of real genus 0, and the group [160,33], which has structure $C_5 \rtimes D_{16}$. This group admits a presentation given by generators A, B, C, D, E, F and relations

$$A^2 = B^2 = E^2 = 1, F^5 = 1, D^2 = E, C^2 = DE, D^{-1}C^{-1}DC = 1, E^{-1}A^{-1}EA = 1,$$
$$E^{-1}B^{-1}EB = 1, E^{-1}C^{-1}EC = 1, E^{-1}D^{-1}ED = 1, F^{-1}B^{-1}FB = 1, F^{-1}C^{-1}FC = 1,$$
$$F^{-1}D^{-1}FD = 1, F^{-1}E^{-1}FE = 1, B^{-1}A^{-1}BA = C, C^{-1}A^{-1}CA = D, C^{-1}B^{-1}CB = D,$$
$$D^{-1}A^{-1}DA = E, D^{-1}B^{-1}DB = E, F^{-1}A^{-1}FA = F^3.$$

It is clear that A, B generate C; A, C generate D; and A, D generate E. Hence the group is generated by A, B, F. Besides, $F^{-1}B^{-1}FB = 1$, and so $FB = BF$. Therefore $(BF)^n = B^nF^n$. Since B has order 2, and F has order 5, we have that $(BF)^5 = B, (BF)^6 = F$, and BF has order 10. In this way, A of order 2, and BF of order 10, generate the group. Hence we can take an NEC group Λ with signature $(0, +, [2,10], \{(-)\})$, and define the epimorphism

$$\theta : \Lambda = (0, +, [2,10], \{(-)\}) \to G$$

by

$$\theta(x_1) = A,\ \theta(x_2) = BF,\ \theta(e_1) = (ABF)^{-1},\ \theta(c_{1,0}) = 1.$$

Since $|\Lambda|^* = \frac{2}{5}$, we have that $\rho(G) \leq 1 + \frac{2}{5} \cdot 160 = 65$.

$$N = 168$$

Now, the group must contain D_{12} and D_{14}. There are just two groups, D_{84}, of real genus 0, and the group [168,17], which has structure $C_3 \rtimes ((C_{14} \times C_2) \rtimes C_2)$. This group has a presentation given by generators A, B, C, D, E, and relations

$$B^2 = C^2 = 1, A^2C^{-1} = 1, D^3 = 1, C^{-1}A^{-1}CA = 1, C^{-1}B^{-1}CB = 1, D^{-1}A^{-1}DA = 1,$$
$$D^{-1}C^{-1}DC = 1, E^{-1}B^{-1}EB = 1, E^{-1}C^{-1}EC = 1, E^{-1}D^{-1}ED = 1, B^{-1}A^{-1}BAC^{-1} = 1,$$
$$D^{-1}B^{-1}DBD^{-1} = 1, E^7 = 1, E^{-1}A^{-1}EAE^{-5} = 1.$$

Since $C = A^2$, the generator C is redundant, and A has order 4. Therefore the group G is generated by A, B, D, E. Besides, $D^{-1}A^{-1}DA = 1$, and so $DA = AD$, the element AD has order 12, and $(AD)^4 = D, (AD)^9 = A$. In the same way, $E^{-1}B^{-1}EB = 1$, and hence $EB = BE$, and we have that BE has order 14, and $(BE)^7 = B, (BE)^8 = E$. We have seen that AD and BE generate the group G. We are going now to prove that the product $(AD)(BE)$ has order 2:

$$ADBEADBE = ADBAE^{-1}DBE = ADABCE^{-1}DBE = AADBCE^{-1}DBE =$$
$$A^2DBCE^{-1}DBE = CDBCE^{-1}DBE = DCBDE^{-1}DBE = DBE^{-1}DBE = DBDE^{-1}BE =$$
$$BE^{-1}BE = BBE^{-1}E = 1.$$

Thus, there are two elements of order 2 and 12, $(AD)(BE)$ and AD, generating G. Hence we can take an NEC group Λ with signature $(0, +, [2,12], \{(-)\})$, and define the epimorphism

$$\theta : \Lambda = (0, +, [2,12], \{(-)\}) \to G$$

by

$$\theta(x_1) = (AD)(BE),\ \theta(x_2) = AD,\ \theta(e_1) = (ADBEAD)^{-1},\ \theta(c_{1,0}) = 1.$$

Since $|\Lambda|^* = \frac{5}{12}$, we have that $\rho(G) \leq 1 + \frac{5}{12} \cdot 168 = 71$.

$$N = 180$$

The group must contain D_9 and D_{10}. There are just two groups, D_{90}, of real genus 0, and $D_5 \times D_9$, of real genus 37.

$$N = 200$$

In this last order, the group must contain D_4 and D_{25}. The two groups satisfying that condition are D_{100}, of real genus 0, and the group [200,8], which has structure $C_{25} \rtimes C_8$. This group has a presentation given by generators $A. B, C, D, E$, and relations

$$A^2 = B^2 = C^2 = 1, C^{-1}A^{-1}CA = 1, C^{-1}B^{-1}CB = 1, D^{-1}B^{-1}DB = 1, D^{-1}C^{-1}DC = 1,$$
$$E^{-1}B^{-1}EB = 1, E^{-1}C^{-1}EC = 1, E^{-1}D^{-1}ED = 1, E^5 = 1, B^{-1}A^{-1}BAC^{-1} = 1,$$
$$E^{-1}A^{-1}EAE^{-3} = 1, D^{-1}A^{-1}DAE^{-1}D^{-3} = 1, D^5E^{-4} = 1.$$

Then, $D^5 = E^4 = E^{-1}$, and so E is generated by D. Besides, D has order 25. Also, C is generated by A and B. Hence the group G is generated by A, B, D. Furthermore, B and D commute, and so BD has order 50, $(BD)^{25} = B, (BD)^{26} = D$. The group G is generated by A of order 2, and BD of order 50. Now we are going to see that the product $A(BD)$ has order 4, by proving that $(ABD)^2 = C$, of order 2. First of all, observe that

$$1 = D^{-1}A^{-1}DAE^{-1}D^{-3} = D^{-1}A^{-1}DAD^5D^{-3} = D^{-1}A^{-1}DAD^2.$$

Hence, $D = A^{-1}DAD^2$, implying $1 = A^{-1}DAD$, and so $DAD = A$. Besides, $B^{-1}A^{-1}BA = C$, what implies $A^{-1}BA = BC, BA = ABC$, and finally $BAC = AB$. Hence, $ABD = BACD = BADC$, and then

$$(ABD)^2 = (BADC)(ABD) = BADACBD = BADABCD = BADABDC = BADADBC =$$
$$BAABC = BBC = C.$$

We have finished, $A(BD)$ has order 4, and the group G is generated by A of order 2, and ABD of order 4. Hence we can take an NEC group Λ with signature $(0, +, [2,4], \{(-)\})$, and define the epimorphism

$$\theta : \Lambda = (0, +, [2,4], \{(-)\}) \to G$$

by

$$\theta(x_1) = A,\ \theta(x_2) = ABD,\ \theta(e_1) = (BD)^{-1},\ \theta(c_{1,0}) = 1.$$

Since $|\Lambda|^* = \frac{1}{4}$, we have that $\rho(G) \leq 1 + \frac{1}{4} \cdot 200 = 51$.

4. THE RESULT AND CONCLUSIONS

Since we have checked all possible orders for the group we were looking for, the following result has been proved.

Theorem. There exists no group of real genus 72.

Remark. The above result proves that the existence of the already known gaps in the real genus spectrum, 2, 12 and 24, is not just a question of small numbers. So, it is necessary to continue the search for more possible gaps. The first numbers N for which it is not yet known whether they are the real genus of a group are now 84, 108, 168 and 192. The technique in this paper should be useful at least for 84 and 108, although the problems, both computational and on the structure of the groups, grow with N.

Acknowledgements. The results in the present work form part of the Degree Thesis of P. Rodríguez, which was directed by J.J. Etayo, with the collaboration of E. Martínez.

J.J. Etayo is partially supported by the Projects UCM910444 and PID2021-122752NB-I00.

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